**5.1. Sectional curvature of submanifolds.** Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with sectional curvature $\bar{\sec}$. Let $p \in \bar{M}$ and $L \subset T\bar{M}_p$ an $m$-dimensional linear subspace.

1. Prove that there is some $r > 0$ such that for the open ball $B_r(0) \subset T\bar{M}_p$, the set $M := \exp_p(L \cap B_r(0))$ is an $m$-dimensional submanifold of $\bar{M}$.
2. Let $g$ be the induced metric on $M$ and let $\sec$ be the sectional curvature of $M$. Show that for $E \subset TM_p$, we have $\sec_p(E) = \bar{\sec}_p(E)$ and if $L$ is a 2-dimensional subspace, then $\sec \leq \bar{\sec}$ on $M$.

**5.2. Codazzi equation.** Let $M \subset \bar{M}$ be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. For the second fundamental form $h$ of $M$, we define

$$(D_X^\perp h)(Y, W) := (\bar{D}_X(h(Y, W))^\perp - h(D_X Y, W) - h(Y, D_X W),$$

where $W, X, Y \in \Gamma(TM)$. Show that the Codazzi equation

$$\left(\bar{R}(X, Y)W\right)^\perp = (D_X^\perp h)(Y, W) - (D_Y^\perp h)(X, W)$$

holds for all $W, X, Y \in \Gamma(TM)$.

**5.3. Asymptotic expansion of the circumference.** Let $M$ be a manifold, $E \subset TM_p$ a linear 2-plane and $\gamma_r \subset E$ a circle with center 0 and radius $r > 0$ sufficiently small. Show that

$$L(\exp(\gamma_r)) = 2\pi \left( r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right)$$

for $r \to 0$.